Monotone Comparative Statics

Econ 3030 Fall 2025

Lecture 7

Outline

1. Comparative Statics Without Calculus
2. Supermodularity and Single Crossing
3. Topkis and Milgrom & Shannon's Theorems
4. Finite Data

Announcement

The midterm will be on October 9.

Comparative Statics Without Calculus: Introduction

- Consider a function $f(x, q)$ where x and q are real numbers.
- Assume $f_{xx}(x, q) < 0$ (this is $\frac{\partial^2 f(x,q)}{\partial x \partial x}$, the second derivative w.r.t. x)
- We want to solve

$$\max f(x, q), \text{ subject to } q \in \Theta, x \in S(q)$$

- Let

$$x^*(q) = \arg \max_{q \in \Theta,\ x \in S(q)} f(x, q)$$

- What do we know about how $x^*(q)$ changes with q?
- Use the implicit function theorem

Comparative Statics Without Calculus

Remark

- How does $x^*(q) = \arg\max_{q\in\Theta,\ x\in S(q)} f(x,q)$ change with q?
- Using the implicit function theorem, one can show that if there are complementarities between the choice variable x and the parameter q, the optimum increases in q.

First Order Condition: $f_x(x,q) = 0$. **Second Order Condition**: $f_{xx}(x,q) < 0$.

IFT:

$$x_q^*(q) = -\frac{f_{xq}(x,q)}{f_{xx}(x,q)}$$

Then

$$x_q^*(q) \geq 0 \text{ if and only if } f_{xq}(x,q) \geq 0$$

Issues with implicit function theorem:

1. IFT needs calculus.
2. Conclusions hold only in a neighborhood of the optimum.
3. Results are dependent on the functional form used for the objective function.
    1. In particular, IFT gives cardinal results that depend on the assumptions on f.

Monotone Comparative Statics

Objectives

- Monotone Comparative Statics give results about "changes" that:
  - do not need calculus;
  - are not necessarily local;
  - are ordinal (that is, robust to monotonic transformations);
  - allow for non-uniqueness of the optimum.

- One can get conclusions similar to IFT without calculus.
- The downside is that the results are not as strong.

Main Idea: Complementarities

- The central idea generalizes the notion of complementarities between endogenous variable and parameters.
  - With calculus, this is the assumption $f_{xq}(x, q) \geq 0$.

- If the optimum is not unique, then $x^*(q)$ is a correspondence, but what does it mean for a correspondence to be increasing?

Strong Set Order

- Ranking real numbers is easy, but how can we express the idea that one set is bigger than another set?

Definition

For two sets of real numbers A and B, define the binary relation $\geq_s$ as follows:

$$\boxed{A \geq_s B \quad \text{if} \quad \begin{array}{c} \text{for any } a \in A \text{ and } b \in B \\ \min\{a, b\} \in B \quad \text{and} \quad \max\{a, b\} \in A \end{array}}$$

$A \geq_s B$ reads "A is greater than or equal to B in the strong set order".

- Generalizes the notion of greater than from numbers to sets of numbers.
- This definition reduces to the standard definition when sets are singletons.

Example

Suppose $A = \{1, 3\}$ and $B = \{0, 2\}$. Then, A is not greater than or equal to B in the strong set order.

Non-Decreasing Correspondences

Definition

We say a correspondence $g : \mathbb{R}^n \to 2^{\mathbb{R}}$ is non-decreasing in $\mathbf{x}$ if and only if

$$\mathbf{x}' > \mathbf{x} \qquad \text{implies} \qquad g(\mathbf{x}') \geq_s g(\mathbf{x})$$

- Thus, $\mathbf{x}' > \mathbf{x}$ implies that for any $y' \in g(\mathbf{x}')$ and $y \in g(\mathbf{x})$: $\min\{y', y\} \in g(\mathbf{x})$ and $\max\{y', y\} \in g(\mathbf{x}')$.
  - Larger points in the domain correspond to larger sets in the codomain.
- Generalizes the notion of increasing function to correspondences.

Monotone Comparative Statics: Simplest Case

Set up

- Suppose some function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is the objective function; this is not necessarily concave or differentiable, and the optimizer could be set valued.

- Let
$$x^*(q) = \arg\max f(x, q), \text{ subject to } q \in \Theta, \ x \in S(q)$$
  - Note: for any strictly increasing h, this problem is equivalent to
$$x^*(q) = \arg\max h(f(x, q)), \text{ subject to } q \in \Theta, \ x \in S(q)$$
    - $h(\cdot)$ may destroy smoothness or concavity properties of the objective function so IFT may not work.
- For now, assume $S(\cdot)$ is independent of q (no constraints).
- Assume existence of a solution, but not uniqueness.

Supermodularity

Definition

The function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is supermodular in (x, q) if

$$\text{for all } x' > x \qquad f(x', q) - f(x, q) \text{ is non-decreasing in } q.$$

- If f is supermodular in (x, q), then the incremental gain to a higher x is greater when q is higher.
- This is the idea that x and q are "complements".

Supermodularity

Definition

The function $f : \mathbb{R}^2 \to \mathbb{R}$ is supermodular in (x, q) if

$$\text{for all } x' > x \qquad f(x', q) - f(x, q) \text{ is non-decreasing in } q.$$

- Non decreasing in q means

$$q' > q \Rightarrow f(x', q') - f(x, q') \geq f(x', q) - f(x, q)$$

Question 1, Problem Set 4.

Show that supermodularity is equivalent to the property that

$$q' > q \qquad \text{implies} \qquad f(x, q') - f(x, q) \text{ is non-decreasing in } x$$

Differentiable Version of Supermodularity

- When f is smooth, supermodularity has a characterization in terms of derivatives.

Lemma

A twice continuously differentiable function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is supermodular in (x, q) if and only if $\frac{\partial^2 f(x,q)}{\partial x \partial q} \geq 0$ for all (x, q).

- The inequality in the definition of supermodularity is just the discrete version of the mixed-partial condition in the lemma.
  - $q' > q$ implies $f(x, q') - f(x, q)$ is non-decreasing in x

Topkis' Monotonicity Theorem

Theorem (Easy Topkis' Monotonicity Theorem)

If f is supermodular in (x, q), then $x^*(q) = \arg\max f(x, q)$ is non-decreasing.

Proof.

Let $q' > q$ and take $x \in x^*(q)$ and $x' \in x^*(q')$. We need to show $x^*(q') \geq_s x^*(q)$.

- First show that $\max\{x, x'\} \in x^*(q')$
  - $x \in x^*(q)$ implies $f(x, q) - f(\min\{x, x'\}, q) \geq 0$.
  - $x \in x^*(q)$ also implies that $f(\max\{x, x'\}, q) - f(x', q) \geq 0$
    - verify these by checking two cases, $x > x'$ and $x' > x$.
  - By supermodularity, $f(\max\{x, x'\}, q') - f(x', q') \geq 0$,
  - Hence $\max\{x, x'\} \in x^*(q')$.
- Now show that $\min\{x, x'\} \in x^*(q)$
  - $x' \in x^*(q')$ implies that $f(x', q') - f(\max\{x, x'\}, q) \geq 0$ (by supermodularity),
    - or equivalently $f(\max\{x, x'\}, q) - f(x', q') \leq 0$.
  - $\max\{x, x'\} \in x^*(q')$ also implies that $f(\max\{x, x'\}, q') - f(x', q') \geq 0$,
  - which by supermodularity implies $f(x, q) - f(\min\{x, x'\}, q) \leq 0$
  - Hence $\min\{x, x'\} \in x^*(q)$.

□

Topkis' Monotonicity Theorem

Theorem (Easy Topkis' Monotonicity Theorem)

If f is supermodular in (x, q), then $x^*(q) = \arg\max f(x, q)$ is non-decreasing.

- Supermodularity is sufficient to draw comparative statics conclusions in optimization problems without other assumptons.
- Topkis' Theorem follows from the IFT whenever the standard full-rank condition in the IFT holds.
  - At a maximum, if $f_{xx}(x, q) \neq 0$, it must be negative (by the second-order condition), hence the IFT tells you that $x^*(q)$ is strictly increasing.
    - Remember, IFT says: $x_q^*(q) = -\frac{f_{xq}(x,q)}{f_{xx}(x,q)}$

Example

Profit Maximization Without Calculus

- A monopolist chooses output q to solve $\max p(q)q - c(q,\mu)$.
  - $p(\cdot)$ is the demand (price) function
  - $c(\cdot)$ is the cost function
    - costs depend on the existing technology, described by some parameter μ.
- Let $q^*(\mu)$ be the monopolist's optimal quantity.
- Suppose $-c(q,\mu)$ is supermodular in (q,μ); then the entire objective function is also supermodular in (q,μ).
  - this follows because the first term of the objective does not depend on μ.
- Notice that supermodularity says that for all $q' > q$, $-c(q',\mu) + c(q,\mu)$ is nondecreasing in μ.
  - in other words, the marginal cost is decreasing in μ.
- Conclusion: by Topkis' theorem q^* is nondecreasing as long as the marginal cost of production decreases in the technological progress parameter μ.

Single-Crossing

- In constrained maximization problems, $x \in S(q)$, supermodularity is not enough for Topkis' theorem.

Definition

The function $f : \mathbb{R}^2 \to \mathbb{R}$ satisfies single-crossing in (x, q) if for all $x' > x$, $q' > q$

$$f(x', q) - f(x, q) \geq 0 \qquad \text{implies} \qquad f(x', q') - f(x, q') \geq 0$$

and

$$f(x', q) - f(x, q) > 0 \qquad \text{implies} \qquad f(x', q') - f(x, q') > 0.$$

- As a function of the second argument, the 'marginal return' can cross 0 at most once; whenever it crosses 0, as the second argument continues to increase, the marginal return will remain positive.

Theorem

If f satisfies single crossing in (x, q), then $x^*(q) = \arg\max_{x \in S(q)} f(x, q)$ is nondecreasing. Moreover, if $x^*(q)$ is nondecreasing in q for all constraint choice sets S, then f satisfies single-crossing in (x, q).

- Next, we generalize to higher dimensions.

Definitions

Suppose $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$.

- The join of $\mathbf{x}$ and $\mathbf{y}$ is defined by
$$\mathbf{x} \vee \mathbf{y} = (\max\{x_1, y_1\}, \max\{x_2, y_2\}, \ldots, \max\{x_n, y_n\}).$$
- The meet of $\mathbf{x}$ and $\mathbf{y}$ is defined by
$$\mathbf{x} \wedge \mathbf{y} = (\min\{x_1, y_1\}, \min\{x_2, y_2\}, \ldots, \min\{x_n, y_n\}).$$

- Draw a picture.

Strong Set Order

- We generalize the strong set order definition to $\mathbb{R}^n$.

Definition (Strong set order in $\mathbf{R}^n$)

The binary relation $\geq_s$ is defined as follows: for $A, B \subset \mathbb{R}^n$,

$$A \geq_s B \quad \text{if} \quad \begin{array}{c} \text{for any } \mathbf{a} \in A \text{ and } \mathbf{b} \in B \\ \mathbf{a} \wedge \mathbf{b} \in B \quad \text{and} \quad \mathbf{a} \vee \mathbf{b} \in A \end{array}$$

- The meet is in the smaller set, while the join is in the larger set.
- One uses this to talk about non-decreasing $\mathbb{R}^n$-valued correspondences.

- We look at functions $f(\mathbf{x}, \mathbf{q})$ where the first argument represents the endogenous variables and the second represents the parameters.

Quasi-Supermodularity

Definition

The function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is quasi-supermodular in its first argument if, for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\mathbf{q} \in \mathbb{R}^m$:

1. $f(\mathbf{x}, \mathbf{q}) - f(\mathbf{x} \wedge \mathbf{y}, \mathbf{q}) \geq 0 \quad \Rightarrow \quad f(\mathbf{x} \vee \mathbf{y}, \mathbf{q}) - f(\mathbf{y}, \mathbf{q}) \geq 0;$
2. $f(\mathbf{x}, \mathbf{q}) - f(\mathbf{x} \wedge \mathbf{y}, \mathbf{q}) > 0 \quad \Rightarrow \quad f(\mathbf{x} \vee \mathbf{y}, \mathbf{q}) - f(\mathbf{y}, \mathbf{q}) > 0.$

- This generalizes the 'mixed' second partial derivatives typically used to make statements about complementarities.
- Quasi-supermodularity is an ordinal property (robust to strictly increasing transformations)
- For differentiable functions there is a sufficient condition for quasi-supermodularity.

Exercise

Show that if $f(\mathbf{x}, \mathbf{q})$ is twice differentiable in $\mathbf{x}$ and $\frac{\partial^2 f}{\partial x_i \partial x_j} > 0$ for all $i, j = 1, \ldots, n$ with $i \neq j$ then f is quasi-supermodular in $\mathbf{x}$.

Single-Crossing Property

Definition

The function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ satisfies the single-crossing property if, for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\mathbf{q}, \mathbf{r} \in \mathbb{R}^m$ such that $\mathbf{x} \geq \mathbf{y}$ and $\mathbf{q} \geq \mathbf{r}$:

1. $f(\mathbf{x}, \mathbf{r}) - f(\mathbf{y}, \mathbf{r}) \geq 0 \quad \Rightarrow \quad f(\mathbf{x}, \mathbf{q}) - f(\mathbf{y}, \mathbf{q}) \geq 0;$
2. $f(\mathbf{x}, \mathbf{r}) - f(\mathbf{y}, \mathbf{r}) > 0 \quad \Rightarrow \quad f(\mathbf{x}, \mathbf{q}) - f(\mathbf{y}, \mathbf{q}) > 0.$

- The "marginal return" $f(\mathbf{x}, \cdot) - f(\mathbf{y}, \cdot)$ as a function of the second argument can cross 0 at most once.
- The single-crossing property is an ordinal property (robust to strictly increasing transformations)
- For differentiable functions there is a sufficient condition for single-crossing.

Exercise

Show that if $f(\mathbf{x}, \mathbf{q})$ is twice differentiable and $\frac{\partial^2 f}{\partial x_i \partial q_k} > 0$ for all $i = 1, \ldots, n$ and $k = 1, \ldots, m$ then f satisfies the single-crossing property.

Monotone Comparative Statics

Theorem (easy Milgrom and Shannon)

Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$. *Define* $x^*(\mathbf{q}) = \arg\max_{x \in \mathbb{R}^n} f(\mathbf{x}, \mathbf{q})$. *Suppose* $|x^*(\mathbf{q})| = 1$ *for all* $\mathbf{q}$ *and* $f(\mathbf{x}, \mathbf{q})$ *is quasi-supermodular in its first argument and satisfies the single-crossing property. Then*

$$\mathbf{q} \geq \mathbf{r} \Rightarrow x^*(\mathbf{q}) \geq x^*(\mathbf{r}).$$

- 'Easy' because it assumes the optimum is unique (thus, the proof does not use 'strict' quasi-supermodularity and single-crossing).

Proof.

Suppose $\mathbf{q} \geq \mathbf{r}$. Then:

$$\begin{aligned}
& f\left(x^*(\mathbf{r}), \mathbf{r}\right) \geq f\left(x^*(\mathbf{q}) \wedge x^*(\mathbf{r}), \mathbf{r}\right) && \text{by definition of } x^*(\mathbf{q}) \\
\Rightarrow \quad & f\left(x^*(\mathbf{q}) \vee x^*(\mathbf{r}), \mathbf{r}\right) \geq f(x^*(\mathbf{q}), \mathbf{r}) && \text{by quasi-supermodularity in } \mathbf{x} \\
\Rightarrow \quad & f\left(x^*(\mathbf{q}) \vee x^*(\mathbf{r}), \mathbf{q}\right) \geq f(x^*(\mathbf{q}), \mathbf{q}) && \text{by Single Crossing} \\
\Rightarrow \quad & x^*(\mathbf{q}) \vee x^*(\mathbf{r}) = x^*(\mathbf{q}) && \text{since } |x^*(\mathbf{q})| \text{ equals } 1 \\
\Rightarrow \quad & x^*(\mathbf{q}) \geq x^*(\mathbf{r}) && \text{by Question 2, PS4}
\end{aligned}$$

□

- This result can be extended to constrained optimization problems and to multi-valued optimizers (see Milgrom & Shannon).

Demand Data and Rationality: Motivation

Main Idea

- We observe finite data and want to know if it could have been the result of rational behavior (i.e. maximizing a preference relation or a utility function).

Demand data observations

- We observe N consumption choices made by an individual, given her income and prices (also observable):

$$\mathbf{x}^1, \mathbf{p}^1, w^1 \qquad \mathbf{x}^2, \mathbf{p}^2, w^2 \qquad \mathbf{x}^3, \mathbf{p}^3, w^3 \qquad \ldots \qquad \mathbf{x}^N, \mathbf{p}^N, w^N$$

  These satisfy:
  - $(\mathbf{x}^j, \mathbf{p}^j, w^j) \in \mathbb{R}^n_+ \times \mathbb{R}^n_{++} \times \mathbb{R}_+$ for all $j = 1, .., N$; and
  - $\mathbf{p}^j \cdot \mathbf{x}^j \leq w^j$ for all $j = 1, .., N$.
- In other words, we have finitely many observations on behavior.

- What conditions must these observations satisfy for us to conclude they are the result of the maximizing of a preference relation or a utility function?
- Answer: Something similar to, but stronger than, revealed preference.

An Example (from Kreps)

- There are 3 goods; a choice, given income w and prices $(p_1 p_2, p_3)$, is $(x_1 x_2, x_3)$
- We observe the following:

	w	$\mathbf{p}$	$\mathbf{x}$
observation 1	300	$(10, 10, 10)$	$(10, 10, 10)$
observation 2	130	$(10, 1, 2)$	$(9, 25, 7.5)$
observation 3	110	$(1, 1, 10)$	$(15, 5, 9)$

- Are these choices consistent with rational behavior? Is there a preference/utility function that generates these choices?
  - Sure: suppose the consumer strictly prefers $(500, 500, 500)$ to anything else, and is indifferent among all other bundles.
  - Since $(500, 500, 500)$ is never affordable, any other choice is rationalizable.
    - This seems silly, and not something that should worry us since we can never rule it out.
- So, we start by assuming local non satiation.

Consequences of Local Non Satiation

- The following is slightly different from Full Expenditure.

Lemma

Suppose $\succsim$ is locally non-satiated, and let $\mathbf{x}^*$ be an element of Walrasian demand (therefore $\mathbf{x}^* \succsim \mathbf{x}$ for all $\mathbf{x} \in \{\mathbf{x} \in X : \mathbf{p} \cdot \mathbf{x} \leq w\}$). Then

$$\mathbf{x}^* \succsim \mathbf{x} \qquad \textit{when} \qquad \mathbf{p} \cdot \mathbf{x} = w$$

and

$$\mathbf{x}^* \succ \mathbf{x} \qquad \textit{when} \qquad \mathbf{p} \cdot \mathbf{x} < w$$

- The maximal bundle is weakly preferred to any bundle that costs the same.
- The maximal bundle is strictly preferred to any bundle that costs less.

Proof.

The first part is immediate, by definition of Walrasian demand.
For the second part note that if $\mathbf{p} \cdot \mathbf{x} < w$ then by local non satiation and continuity of $\mathbf{p} \cdot \mathbf{x}$ there exists some $\mathbf{x}'$ such that $\mathbf{x}' \succ \mathbf{x}$ and $\mathbf{p} \cdot \mathbf{x}' \leq w$.
Thus $\mathbf{x}'$ is affordable and $\mathbf{x}^* \succsim \mathbf{x}' \succ \mathbf{x}$ as desired. $\square$

Back to the Example

- We observe the following:

	w	$\mathbf{p}$	$\mathbf{x}$
observation 1	300	$(10, 10, 10)$	$(10, 10, 10)$
observation 2	130	$(10, 1, 2)$	$(9, 25, 7.5)$
observation 3	110	$(1, 1, 10)$	$(15, 5, 9)$

- The consumer spends her entire income in all cases
  - this is consistent with local non satiation.

Furthermore:

- At prices $(10, 10, 10)$ the bundle $(15, 5, 9)$ could have been chosen (it costs 290).
- At prices $(10, 1, 2)$ the bundle $(10, 10, 10)$ could have been chosen (it costs 130).
- At prices $(1, 1, 10)$ the bundle $(9, 25, 7.5)$ could have been chosen (it costes 109).
- What does all this tell us?

Example Continued

Consumer's choices among 3 goods

	w	p	x
observation 1	300	$(10, 10, 10)$	$(10, 10, 10)$
observation 2	130	$(10, 1, 2)$	$(9, 25, 7.5)$
observation 3	110	$(1, 1, 10)$	$(15, 5, 9)$

- Using local non satiation we can conclude the following.
- Since at prices $(10, 10, 10)$, the bundle $(15, 5, 9)$ costs strictly less than $(10, 10, 10)$:
$$\boxed{(10, 10, 10) \succ (15, 5, 9)}$$
- Since at prices $(10, 1, 2)$, the bundle $(9, 25, 7.5)$ costs the same as $(10, 10, 10)$:
$$\boxed{(9, 25, 7.5) \succsim (10, 10, 10)}$$
- Since at prices $(1, 1, 10)$, the bundle $(9, 25, 7.5)$ costs strictly less than $(15, 5, 9)$:
$$\boxed{(15, 5, 9) \succ (9, 25, 7.5)}$$
- Putting these together:
$$(10, 10, 10) \succ (15, 5, 9) \succ (9, 25, 7.5) \succsim (10, 10, 10)$$
- These observations are not consistent with utility/preference maximization theory!

Example End

Suppose instead the data looks as follows

	w	p	x
observation 1	300	$(10, 10, 10)$	$(10, 10, 10)$
observation 2	130	$(10, 1, 2)$	$(9, 25, 7.5)$
observation $3'$	115	$(1, 2, 10)$	$(15, 5, 9)$

- Since at prices $(10, 10, 10)$, the bundle $(15, 5, 9)$ costs strictly less than $(10, 10, 10)$:
$$(10, 10, 10) \succ (15, 5, 9).$$
- Since at prices $(10, 1, 2)$, the bundle $(9, 25, 7.5)$ costs the same as $(10, 10, 10)$:
$$(9, 25, 7.5) \succsim (10, 10, 10).$$
- At prices $(1, 2, 10)$, no other bundle is affordable thus we no longer have a contradiction.
- In fact, there are two possibilities:
$$(9, 25, 7.5) \succsim (10, 10, 10) \succ (15, 5, 9) \quad \text{or} \quad (9, 25, 7.5) \sim (10, 10, 10) \succ (15, 5, 9)$$

Next Class

- Generalized Axiom of Revealed Preferences
- Afriat's Theorem